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# Surface fractal dimension of two-dimensional percolation

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**Abstract.** We determine the surface fractal dimension $D_s$ for two-dimensional percolation from Monte Carlo data in finite systems. Our results agree with the prediction from conformal invariance, $D_s = \frac{2}{3}$.

In this paper we study percolation on a two-dimensional semi-infinite lattice. In the usual percolation problem (Stauffer 1985), i.e. on an infinite lattice, one occupies lattice sites with a probability $p$, independent for each lattice site. It then turns out that for $p > p_c$ (where $0 < p_c < 1$ in $d \geqslant 2$) there exists one infinite cluster of occupied nearest-neighbour sites. Thus, the probability $P$ that a randomly chosen site belongs to the infinite cluster is strictly positive only for $p > p_c$. Furthermore, if $p$ approaches $p_c$ from above, $P$ will go to zero with a power law

$$P(p) \underset{p \to p_c}{\sim} (p - p_c)^{\beta}. \tag{1}$$

If one breaks the translational invariance in one direction by considering a semi-infinite lattice, $P$ will become a function of the distance $z$ from the surface.

For $p$ sufficiently close to $p_c$ one expects scaling laws to hold and, following Christou and Stinchcombe (1986), we can thus assume $P(p, z)$ to have the following scaling behaviour:

$$P(p, z) \sim (p - p_c)^{\beta} F(\xi / z) \tag{2}$$

where $\xi$ is some suitably defined correlation length which near $p_c$ will diverge as

$$\xi(p) \sim |p - p_c|^{-\nu}. \tag{3}$$

The scaling function $F(x)$ is required to have the following property:

$$F(x) \sim \text{constant for } x \to 0$$

implying that for large $z$ one recovers the behaviour (1). On the other hand, for sites near the surface one can expect that they have a smaller probability for belonging to the infinite cluster, and thus, as usual in surface critical behaviour (Binder 1983), we should have

$$F(x) \sim x^{-\sigma} \qquad \text{for } x \to \infty$$

leading to

$$P(p, z \simeq 0) \sim (p - p_c)^{\beta'} \tag{4}$$

where $\beta' = \beta + \nu\sigma$.

It is well known that a scaling law such as (2) can be derived from an assumption of generalised homogeneity for the free energy.

In percolation the role of free energy is played by the generating function for the cluster numbers $n_s$ (Stauffer 1979). For percolation in a semi-infinite lattice one considers (De 'Bell 1980) the following free energy:

$$f(p, h, h') = \sum_{s, s_0} \exp(-sh - s_0 h') n_{s, s_0} \tag{5}$$

where $n_{s,s_0}$ is the average number of clusters (per site) containing $s$ sites of which $s_0$ are at the surface, and $h$ and $h'$ are bulk and surface 'magnetic' fields.

The free energy (5) can be split into a bulk and a surface part $f_s$ whose singular part behaves as (for $|p - p_c|$, $h$, $h' \to 0$)

$$f_s(p - p_c, h, h') \sim b^{-d+1} f_s(b^{y_t}(p - p_c), b^{y_h} h, b^{y'_h} h') \tag{6}$$

introducing the surface 'magnetic' exponent $y'_h$, which is related to $\beta'$ by

$$\beta' = \nu(d - 1 - y'_h). \tag{7}$$

It is our purpose in this paper to calculate $y'_h$ for $d = 2$.

We first would like to remark that $y'_h$ has the physical interpretation of the fractal dimension (Mandelbrot 1982) of the sites at the surface and in the infinite cluster.

Indeed, for $p$ slightly above $p_c$, the density of these points, which is given by $P(p, z = 0)$ behaves as (using (3) and (4))

$$P(p, z = 0) \sim \xi^{-\beta'/\nu}$$

which from (7) is

$$P(p, z = 0) \sim \xi^{y'_h - (d-1)}. \tag{8}$$

Furthermore, the density $\rho$ of a fractal with fractal dimension $D_s$ embedded in a space of Euclidean dimension $D$ behaves as a function of a typical distance $L$ as

$$\rho \sim L^{D_s - D}. \tag{9}$$

Combining (8) and (9) immediately gives that the fractal dimension $D_s$ of the sites at the surface of the infinite cluster equals $y'_h$. (Such an argument was first given by Stanley (1977) to show that $y_h$ equals the bulk fractal dimension.)

Recently, it was conjectured on the basis of conformal invariance (Cardy 1984) and from a conjecture for the surface magnetic exponent $y'_h(q)$ of the $q$-state Potts model (Vanderzande and Stella 1987) that $y'_h$ for percolation is equal to $\frac{2}{3}$.

So far, this conjecture has received little support. For example, from (7) we have that $\beta'$ should be equal to $\frac{4}{9} = 0.444\ldots$ (because $\nu = \frac{4}{3}$). However, from a Monte Carlo calculation, Watson (1986) found $\beta' = 0.398 \pm 0.005$, whereas Christou and Stinchcombe (1986) found $\beta' = 1.197$ as the best result in a simple real-space renormalisation calculation.

De 'Bell and Lookman (1986) calculated the exponent $\gamma_1$, which is given by

$$\gamma_1 = \nu(1 - d + y_h + y'_h) \tag{10}$$

from exact series expansions on the triangular lattice. They found $\gamma_1 = 2.07$ ($\pm 0.03$) in good agreement with the conjecture $\gamma_1 = \frac{25}{12}$ (using $y_h = \frac{91}{48}$). On the other hand, De 'Bell and Essam (1980) found $\gamma_{1,1} = 0.57 \pm 0.01$, whereas $\gamma_{1,1}$ should be equal to $\nu(-d + 1 + 2y'_h) = \frac{4}{9}$.

We therefore found it appropriate to have an independent calculation of $y'_h$. We followed the same strategy as we already successfully applied to the Potts model

(Vanderzande and Stella 1987) and the $Z_5$ model (Vanderzande 1987). Here we calculate the surface susceptibility $\chi_s$ which is defined as

$$\chi_s(p) \equiv \left.\frac{\partial^2 f}{\partial h\, \partial h'}\right|_{h=h'=0}. \tag{11}$$

At the infinite systems' percolation threshold $p = p_c$, finite-size scaling (Barber 1983) predicts that the surface susceptibility $\chi_{s,L}$ in a finite system of size $L$ should behave as

$$\chi_{s,L}(p_c) \sim L^{-d+y_h+y'_h} \tag{12}$$

which for two-dimensional percolation gives

$$\chi_{s,L}(p_c) \sim L^{9/16=0.5625}. \tag{13}$$

Furthermore, from (5) and the definition (11) we find that the surface susceptibility is given by

$$\chi_s(p) = \sum_{s,s_0} s s_0 n_{s,s_0}. \tag{14}$$

Our calculations were performed on the square lattice, where $p_c$ is known to be equal to 0.592 75 (Stauffer 1985). We considered finite square systems of side $L$, with $L = 2, 4, 6, \ldots, 20$. We took periodic boundary conditions in one direction, thus creating two free surfaces in the perpendicular direction. For the largest system sizes we generated up to 60 000 configurations, from which (14) was calculated using standard cluster counting methods (Stauffer 1985). Errors were estimated from fluctuations in subresults.

Our results are shown in figure 1 where we plot log $\chi_{s,L}(p_c)$ against log $L$. The data for $L > 6$ lie almost on a perfect straight line whose slope (determined by a least squares fit) is given by $0.57 \pm 0.01$. This is in very nice agreement with the prediction in (13).

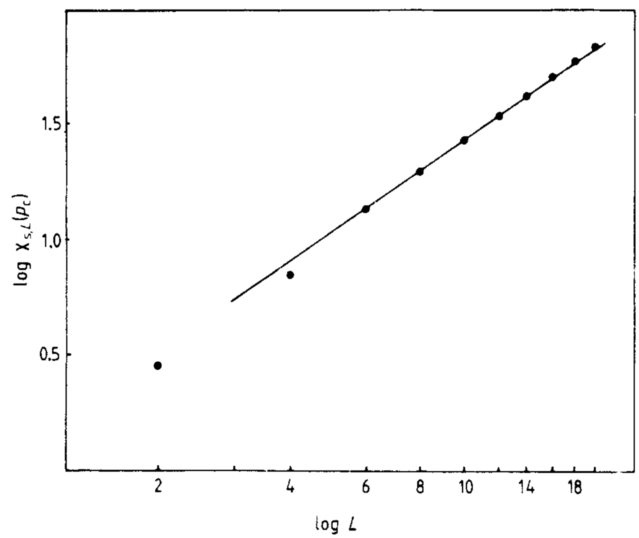


**Figure 1.** Logarithm of surface susceptibility $\chi_{s,L}$ against logarithm of system size $L$, at the percolation threshold. The dots represent our data, the straight line is a best fit through the data for $L > 6$.

A simultaneous calculation of the bulk susceptibility at the percolation threshold $\chi_L(p_c)$ which is given by

$$\chi_L(p_c) \equiv \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=h'=0} = \sum_{s,s_0} s^2 n_{s,s_0} \tag{15}$$

and which should behave as

$$\chi_L(p_c) \sim L^{-d+2y_h} \tag{16}$$

gave as result $y_h = 1.86 \pm 0.01$, to be compared with $y_h = \frac{91}{48} = 1.896$. Clearly, for the sizes considered one cannot neglect surface effects and therefore one should suspect that $y_h$ cannot be determined as accurately as the combination of surface and bulk properties $-d + y_h + y'_h$.

We did not calculate $y'_h$ by calculating another surface susceptibility $\chi'_{s,L}(p_c)$ given by

$$\chi'_{s,L}(p_c) = \left.\frac{\partial^2 f}{\partial h'^2}\right|_{h=h'=0} = \sum_{s,s_0} s_0^2 n_{s,s_0} \tag{17}$$

behaving in finite systems as

$$\chi'_{s,L}(p_c) \sim L^{-d+2y'_h} \sim L^{-2/3}. \tag{18}$$

Although such a calculation would have the advantage of a direct determination of $y'_h$ only, the quantity $\chi'_{s,L}(p_c)$ does not diverge, but instead converges to an analytic background susceptibility which would make it difficult to determine the exponent with great accuracy.

In conclusion, we have calculated the surface fractal dimension, or surface magnetic exponent, for two-dimensional percolation and have found good evidence in favour of the conjecture $y'_h = D_s = \frac{2}{3}$.

We would finally like to remark that in the case of self-avoiding walks and lattice animals, the role of bulk fractal dimension is played by the 'thermal' exponent $y_t$. One can thus expect that for these problems the surface fractal dimensional equals the surface thermal exponent, which for two-dimensional systems (or for three-dimensional systems at the so-called ordinary transition (Binder 1983)) always equals $-1$ (Burkhardt and Cardy 1987). Of course, then the idea of fractal dimension loses much of its meaning.

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